Direct and inverse approximation theorems of functions of several variables in the spaces S^p

V. V. Savchuk, A. L. Shydlich
Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine
E-mail: `vicsavchuk@gmail.com, shidlich@gmail.com`

Let $L_1 = L_1(\mathbb{T}^d)$ be the space of all functions f, given on the d-dimensional torus $\mathbb{T}^d = \prod_{j=1}^d[-\pi,\pi]$ with the usual norm $\|f\|_{L_1}$.

The space S^p, $1 \le p < \infty$, [1, Chap. XI] (see also [2]) is the space of all functions $f \in L_1$ such that $\|f\|_{S^p} := \big(\sum_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{f}(\mathbf{k})|^p\big)^{1/p} < \infty$, where $\widehat{f}(\mathbf{k})$ are the Fourier coefficients of the function f. Functions $f \in L_1$ and $g \in L_1$ are equal in the space S^p, if $\|f-g\|_{S^p} = 0$.

The modulus of continuity of $f \in S^p$ of index $\alpha > 0$ is defined by

$$\omega_\alpha(f,t)_{S^p} = \sup_{|h|\le t} \|\Delta_h^\alpha f\|_{S^p} = \sup_{|h|\le t} \Big\| \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(\mathbf{x} - jh) \Big\|_{S^p}, \quad t > 0,$$

where $\binom{\alpha}{j} = \alpha(\alpha-1)\cdot\ldots\cdot(\alpha-j+1)/j!$, $(\mathbf{x}-jh) := (x_1 - jh, \ldots, x_d - jh)$.

Denote by

$$E_n^{\triangle}(f)_{S^p} = \inf_{a_{\mathbf{k}}\in\mathbb{C}} \|f - \sum_{\nu=0}^{n} \sum_{|\mathbf{k}|_1=\nu} a_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}\|_{S^p}, \quad |\mathbf{k}|_1 := \sum_{j=1}^{d} |k_j|$$

the best approximation of the function $f \in S^p$ by the triangular polynomials of the order $n-1$.

Theorem. *Assume that $f \in S^p$, $1 \le p < \infty$. Then for any $n \in \mathbb{N}$ and $\alpha > 0$, the following relation is true:*

$$\omega_\alpha\Big(f, \frac{\pi}{n}\Big)_{S^p} \le \frac{\pi^\alpha}{n^\alpha} \Bigg(\sum_{\nu=1}^{n} (\nu^{\alpha p} - (\nu-1)^{\alpha p}) E_\nu^{\triangle}(f)_{S^p}^p \Bigg)^{1/p}.$$

In the case of approximation of functions of one variable, for modulus of continuity of integer index, the corresponding result was obtained in [2].

[1] *Stepanets A. I.* Methods of approximation theory. VSP, Leiden (2005), 919 pp.

[2] *Stepanets A. I., Serdyuk A. S.* Direct and inverse theorems in the theory of approximation of functions in the space S^p, Ukr. Math. Journ., 2002, Vol. 54, № 1, pp. 126-148.